Selberg 不等式和筛法

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摘要

本报告将介绍 Selberg 一些原始的想法. 这些想法对现代数论有巨大的影响. 报告面向所有学生，无需很多的预备知识.

Today's lecture will explore how an original idea is produced in the field of analytic number theory.

1 Notations

We first introduce the following notations that will be used throughout the lecture. Denote distinct primes by $p_1, p_2, \cdots$ and natural logarithm by log. Finally, the power function n^s is understood as $e^{(\log n)s}$, which also extends to an arbitrary complex power s.

2 Selberg Inequality

How many primes are there? Infinitely many? The proof goes back to Euclid more than 2,300 years ago, and it's by contradiction.

Theorem 2.1 (Euclid). *There are infinitely many prime numbers.*

But what exactly is the magnitude of this infinity? The *sieve of Eratosthenes* is an efficient method that gives the number of prime numbers on a fixed interval $[1, x]$. Now, if we introduce the function $\pi(x)$ which denote the number of primes less than x(usually a large number), first suggested by Chebyshev, a natural question arises.

Question 2.2. *Does $\pi(x)$ admit an asymptotic expression as $x \to \infty$?*

The following theorem was conjectured independently by Legendre and Gauss.

Theorem 2.3 (Legendre, Gauss). *(The Prime Number Theorem)* $\lim_{x\to\infty} \pi(x) \Big/ \dfrac{x}{\log x} = 1.$

An approximation was first formulated by Legendre via numerical computations. But Gauss' observation was more profound, which makes use of the Mangoldt function

$$\Lambda(n) = \Lambda_1(n) = \begin{cases} \log p, & n = p^m, m \geq 1, \\ 0, & n \neq p^m. \end{cases}$$

If we consider only the square-free n's, $\Lambda(n)$ can be viewed as a weighted characteristic function for prime numbers.

We know that if n is close to x, for instance, $c_1 x < n < c_2 x$, $\log n \sim \log x$. This leads to the formulation of 2.3 , suggested by Gauss:

$$\lim_{x\to\infty} \frac{1}{x} \left(\sum_{p \leq x} \log p \right) = 1.$$

Remark 2.4. 这里向同学们推荐一本书：《素数定理的初等证明》（*1988*，潘承洞，潘承彪著）.

Russian mathematician Chebyshev was among the first ones who worked on the prime number theorem. He proved that the limit superior and limit inferior were both very close to 1. By applying the binomial expansion, where the middle term has the maximal coefficient, he proved that the limit of $\pi(x) \Big/ \dfrac{x}{\log x}$, if it exists, has to be 1.

It wasn't until our insightful Riemann introduced the Zeta function and laid the foundations for analytic number theory that Hadamard and de la Vallée Poussin were able to prove the prime number theorem in 1896. The proof required tools from analysis (we know that the prime number theorem is equivalent to a certain property of an analytic function).

Question 2.5. *Is there an elementary proof for the prime number theorem?*

In 1921, G. H. Hardy stated that he was doubtful such a proof could be found, saying if one was found "that it is time for the books to be cast aside and for the theory to be rewritten."[1] However, two years following Hardy's death in 1947, Erdös and Selberg found an elementary proof.

We shall now follow Selberg and examine the idea of this proof, with no extensive use of calculus or other advanced analytic approaches.

Note that we do make use of the following facts, which are considered "elementary":

[1] Wikipedia page

Remark 2.6.

$$\sum_{n<x} \frac{1}{n} = \log x + \gamma,$$

where γ is the Euler constant (we have dropped the error term) and

$$\sum_{n<x} \frac{\log n}{n} = \frac{1}{2}\log^2 x + c\log x + \cdots .$$

Definition 2.7. $\Lambda_2(n) = \Lambda(n)\log n + \sum_{n_1 n_2 = n} \Lambda(n_1)\Lambda(n_2)$, *where n is square-free.*

Then

$$\Lambda_2(n) = \begin{cases} \log^2 p, & n = p \\ 2\log p_1 \log p_2, & n = p_1 p_2 \\ 0, & \text{otherwise.} \end{cases}$$

2.3 states that $\sum_{n\le x} \Lambda_1(n) \sim x$, which we cannot prove for the time being. However we have the following

Theorem 2.8 (Selberg Inequality)**.** $\sum_{n<x} \Lambda_2(n) = 2x\log x + O(x)$ *for $x \ge 1$.*

How did Selberg come up with this equality? Note that for sufficiently large x, $\sum_{n\le x} \log \frac{x}{n} = O(x)$, since $\log \frac{x}{n} = O\left(\sqrt{\frac{x}{n}}\right)$. Then what can we say about $\sum_{n\le x} \log^k \frac{x}{n}$, where k is a fixed integer? This naturally leads to the Selberg inequality stated above.

Proposition 2.9. $\Lambda_n = \sum_{d|n} \mu(d)\log \frac{1}{d} = \sum_{d|n} \mu(d)\log \frac{x}{d}$, *where $n > 1$ and μ is the Möbius function.*

The Möbius function only takes on three values:

$$\mu(n) = \begin{cases} 1, & n = 1, \\ (-1)^r, & n = p_1 p_2 \cdots p_r \text{ where } p_1, p_2, \cdots, p_r \text{ are distinct,} \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases}$$

Since $\Lambda_2(n) = \sum_{d|n} \mu(d)\log^2 \dfrac{n}{d} \sim \sum_{d|n} \mu(d)\log^2 \dfrac{x}{d}$, we note that

$$\sum_{n\le x} \Lambda(n) = \sum_{n\le x} \left(\sum_{d|n} \mu(d)\log \frac{x}{d} \right) = \sum_{d\le x} \mu(d)\log \frac{x}{d} \sum_{\substack{n\le x \\ d|n}} 1.$$

Taking into consideration that $\sum_{\substack{n\leq x\\ d|n}} 1 = \frac{x}{d} + O(1)$ and $|\mu(d)| \leq 1$, we conclude that

$$\sum_{n\leq x} \Lambda(n) = \sum_{d\leq x} \frac{\mu(d)}{d} x \log\frac{x}{d} + O(x).$$

Selberg was able to deal with the first term by applying $\log\frac{x}{d} = \sum_{k\leq\frac{x}{d}} \frac{1}{k} - \gamma + \cdots$, and found that the dominant term was exactly x. Unfortunately, the error term is also x. Thus he had merely given a different proof of

$$\sum_{n\leq x} \Lambda(n) = O(x),$$

which is nothing other than a repetition of Chebyshev's previous result.

However, if we raise the power of the logarithm:

$$\sum_{n\leq x} \Lambda_2(n) = \sum_{n\leq x} \left(\sum_{d|n} \mu(d) \log^2\frac{x}{d} \right) + O(x) = \sum_{d\leq x} \frac{\mu(d)}{d} x \log^2\frac{x}{d} + O(x).$$

Substituting with

$$\log^2\frac{x}{d} = 2\sum_{k\leq\frac{x}{d}} \frac{\tau(k)}{k} + c_1 \sum_{k\leq\frac{x}{d}} \frac{1}{k} + c_2 + \cdots + O(\_),$$

where $\tau(k) = \sum_{d|k} 1$ is the totient function, applying convolution relations and making variable substitutions we obtain the Selberg inequality.

We would like to point out Bombieri's doctoral dissertation dealt with other powers of the logarithm, leading to the so-called generalised Selberg inequalities.

3 Connections with the Riemman Zeta Function

The well-known Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}\{s\} > 1.$$

How does the ζ function relate to prime numbers and the Λ functions?

We observe that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)},$$

where $\mathrm{Re}(s) > 1$. We also have

$$\sum_{n=1}^{\infty} \frac{\Lambda_2(n)}{n^s} = \frac{\zeta''(s)}{\zeta(s)} = \left[\frac{\zeta'(s)}{\zeta(s)}\right]' + \left[\frac{\zeta'(s)}{\zeta(s)}\right]^2.$$

This is true for not only the zeta function but also all C^2 functions. In fact, one can verify that for $f \in C^2$, it holds that

$$\frac{f''(s)}{f(s)} = \left[\frac{f'(s)}{f(s)}\right]' + \left[\frac{f'(s)}{f(s)}\right]^2.$$

4 Selberg's Sieve Method

It is a bit difficult to explain what the sieve method is. The original sieve method filters out the composite numbers in a given interval and gives a table of prime numbers. Today sieve methods operate on sets whose definitions involve a number x that tends to infinity.

For example, there are several formulations of the Goldbach Conjecture.[2] One of them can be described as follows: let n be sufficiently large. If the set $S = \{2n-p \mid p < 2n\}$ contains a prime number p', we have $2n = p + p'$ and this gives an even number expressed as the sum of two primes. The Goldbach Conjecture therefore simply asserts that for each n the lower bound of $\#\{$ primes in $S\}$ is *strictly* greater than zero.

Norwegian mathematician Viggo Brun was the first to introduce the following: let $\mathscr{D} \subset [1, 2n]$. For $m < 2n$, consider

$$\sum_{\substack{d|m \\ d\in\mathscr{D}}} \mu(d) \begin{cases} > 0, & m = p, \\ \leq 0, & m \neq p. \end{cases}$$

So we only have to justify that

$$\sum_{p<2n} \left(\sum_{\substack{d|2n-p \\ d\in\mathscr{D}}} \mu(d) \right) > 0.$$

Brun also introduced another set $\mathscr{D}^*$ for $m \in [x, x+\sqrt{x}]$, and considered it together with the sum

$$\sum_{\substack{d|m \\ d\in\mathscr{D}^*}} \mu(d) = \begin{cases} 1, & m = p, \\ \geq 0, & m \neq p. \end{cases}$$

We wish to bound the following:

$$\#\{p \mid p \in [x, x+\sqrt{x}]\} \leq \sum_{m\in[x,x+\sqrt{x}]} \left(\sum_{\substack{d|m \\ d\in\mathscr{D}^*}} \mu(d) \right) \leq ?.$$

[2]For more reference see https://people.maths.bris.ac.uk/ matdb/tcc/SIEVES/sieves.pdf

This is *the Brun sieve*, which is fairly inaccurate.

As we can see, the fundamental goal of sieve theory is to produce upper and lower bound of the number of elements for certain finite sets.

For *Selberg's sieve* we use, instead of $\sum\limits_{\substack{d|m \\ d\in\mathscr{D}^*}} \mu(d)$,

$$\left(\sum_{d|m} \lambda_d\right)^2, \qquad \text{where } \lambda_d \in \mathbb{R}, \lambda_1 = 1.$$

Some famous achievements of the sieve method are from Chen etc.

5 The Twin Prime Conjecture

The characteristic function for prime numbers is defined as

$$\rho(n) = \begin{cases} 1, & n = p, \\ 0, & n \neq p. \end{cases}$$

Given $\{h_1, h_2, \cdots, h_k\}$ where $0 = h_1 < h_2 < \cdots < h_k < 7000,000$, let

$$\begin{aligned} f(n) &= \rho(n+h_1) + \rho(n+h_2) + \cdots + \rho(n+h_k) - 1 \\ &= \begin{cases} -1 \\ 0 \\ > 0 \text{ iff at least two of } \{n+h_i\} \text{ are prime.} \end{cases} \end{aligned}$$

Claim: $\forall x > 0, \exists\, n > x$ such that $f(n) > 0$.

We can turn to an enhanced version where we claim that there exists such $n \in (x, 2x)$.

Let $g(n) \in \mathbb{R}$ such that $\sum\limits_{x<n<2x} f(n)g(n)^2 > 0$. If for each x there is such a function $g(n)$, since $g^2 \geq 0$ and at least one of the summed terms is positive, $f(n) > 0$ and we have proved the claim.

Modern sieve methods are all about suitable choices of the function $g(n)$ above.

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